Outline

- Wave equations in source-free region
- Time-harmonic fields in source-free region
- Inhomogeneous wave equation: wave/light generation

- Wave equations
    - Homogeneous equations in time-domain

Time-domain Maxwell's equations

- Simple medium
  - Linear, homogeneous, isotropic: $\vec{D} = \varepsilon \vec{E}; \quad \vec{B} = \mu \vec{H}$
- Charge-free: $\rho = 0$
- Non-conducting: $\vec{J} = 0$

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{D} = \rho \\ \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \end{cases} \longrightarrow \begin{cases} \nabla \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} & (1) \\ \nabla \cdot \vec{E} = 0 & (2) \\ \nabla \times \vec{H} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} & (3) \\ \nabla \cdot \vec{H} = 0 & (4) \end{cases}$$

Time-domain homogeneous wave equation-(1)

- Take curl of Eq. (1) $\nabla \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t}$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \nabla \times \vec{E} = \cancel{\nabla(\nabla \cdot \vec{E})} - \nabla^2 \vec{E}$$

$$= \nabla \times \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) = -\mu \frac{\partial}{\partial t} \left(\nabla \times \vec{H} \right)$$

$$= -\mu \frac{\partial}{\partial t} \left(\varepsilon \frac{\partial \vec{E}}{\partial t} \right) = -\mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Free-space: $\nabla^2 \vec{E} - \dfrac{1}{c^2} \dfrac{\partial^2 \vec{E}}{\partial t^2} = 0$

Time-domain homogeneous wave equation-(2)

- Take curl of Eq. (3) $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$$

$$= \nabla \times \left(\varepsilon \frac{\partial \vec{E}}{\partial t} \right) = \varepsilon \frac{\partial}{\partial t} \left(\nabla \times \vec{E} \right)$$

$$= \varepsilon \frac{\partial}{\partial t} \left(-\mu \frac{\partial \vec{H}}{\partial t} \right) = -\mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Compared to T-Lines

- High similarities

$$\frac{\partial^2}{\partial z^2} v(z,t) = LC \frac{\partial^2}{\partial t^2} v(z,t)$$

$$\frac{\partial^2}{\partial z^2} i(z,t) = LC \frac{\partial^2}{\partial t^2} i(z,t)$$

$$\nabla^2 \vec{E} - \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

$$\nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

Scalar waves → Vector waves

Velocity: $v_p = \frac{1}{\sqrt{LC}}$

Velocity: $u_p = \frac{1}{\sqrt{\mu\varepsilon}}$

Comments

- We assumed charge-free and current-free: $\rho = 0, \ \vec{J} = 0$
  - These equations only deals how the waves propagate.
  - They do not tell us how the waves are generated.

- We assume a simple medium. If the medium is complicated: (nonlinear, anisotropic, inhomogeneous), then the wave equation will be different.

- Time-harmonic fields (frequency-domain)
  - Wave equation with sinusoidal time functions
  - Helmholtz's equations

Why *time-harmonics*?

- Any periodic (aperiodic) function ➔ superposition of discrete (continuous) sinusoidal functions by Fourier series (integral).
- Maxwell's equations are linear.
  - Sinusoidal sources produce sinusoidal fields of the same frequency in steady state.
  - Total field can be derived by superposition of individual sinusoidal responses.
- Easy to operate if phasors are used: $\vec{A}(\vec{r},t) = \vec{A}(\vec{r})e^{j\omega t}$

$$\frac{\partial}{\partial t} \to j\omega, \quad \int dt \to \frac{1}{j\omega}$$

Scalar to vector phasor notation

- Scalar phasors of voltages & currents are sufficient to describe steady-state response of TX lines:

$$v(z,t) = \mathrm{Re}\left\{V(z)\cdot e^{j\omega t}\right\}$$

$$i(z,t) = \mathrm{Re}\left\{I(z)\cdot e^{j\omega t}\right\}$$

- Vector phasors of E-field and M-field are required to describe time-harmonic EM fields:

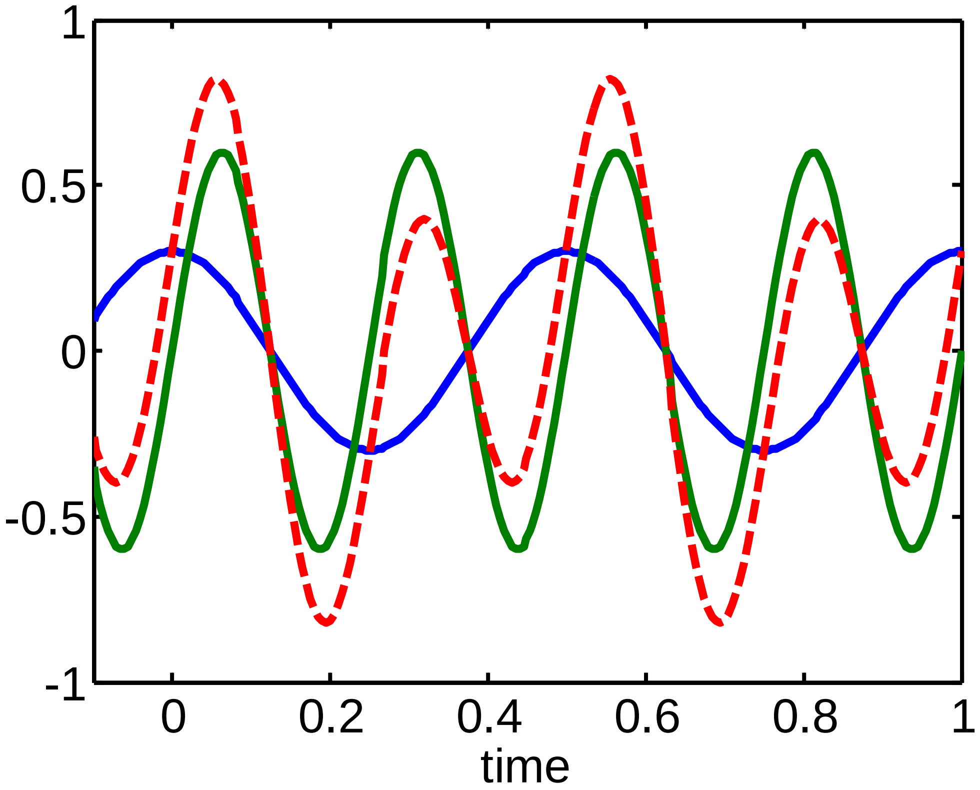
$$\vec{E}(x,y,z,t) = \mathrm{Re}\left\{\vec{E}(x,y,z)e^{j\omega t}\right\}$$

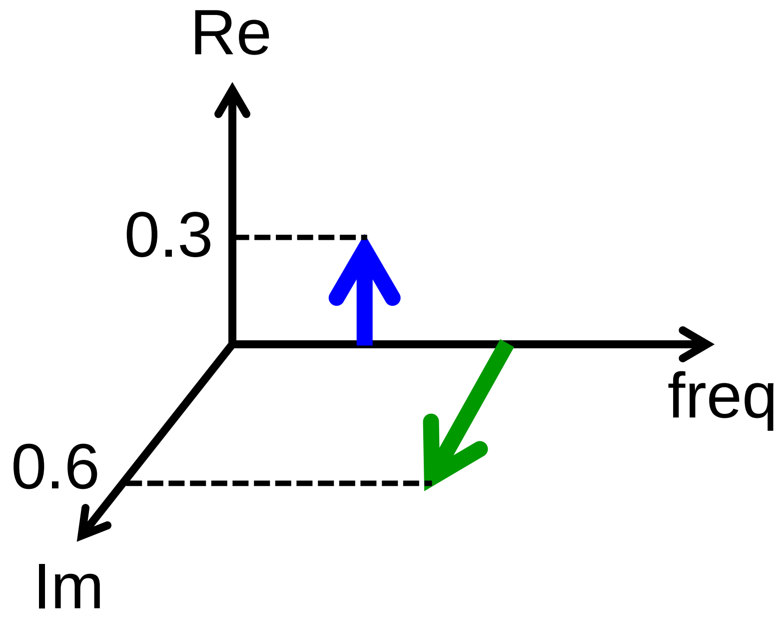
$$\vec{H}(x,y,z,t) = \mathrm{Re}\left\{\vec{H}(x,y,z)e^{j\omega t}\right\}$$

Phasor: time ➔ frequency spectrum!

- Magnitude and phase of a single-frequency

$$e(t) = 0.3\cos(2\pi \times 2t) + 0.6\sin(2\pi \times 4t)$$





Frequency-domain Maxwell's equations

- For simple, source-free, current-free medium

$$\begin{cases} \nabla \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} \\ \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{H} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} \\ \nabla \cdot \vec{H} = 0 \end{cases} \xrightarrow[\frac{\partial}{\partial t} \to j\omega]{\vec{E}(\vec{r},t) \to \vec{E}(\vec{r}),\ \vec{H}(\vec{r},t) \to \vec{H}(\vec{r})} \begin{cases} \nabla \times \vec{E} = -j\omega\mu\vec{H} & (1) \\ \nabla \cdot \vec{E} = 0 & (2) \\ \nabla \times \vec{H} = j\omega\varepsilon\vec{E} & (3) \\ \nabla \cdot \vec{H} = 0 & (4) \end{cases}$$

Frequency-domain wave equation-(1)

- Take curl of Eq. (1) $\nabla\times\vec{E}=-j\omega\mu\vec{H}$

$$\nabla\times\nabla\times\vec{E}=\nabla\left(\nabla\cdot\vec{E}\right)-\nabla^2\vec{E}=-\nabla^2\vec{E}$$

$$\nabla\times\vec{H}=j\omega\varepsilon\vec{E}$$

$$=\nabla\times\left(-j\omega\mu\vec{H}\right)=-j\omega\mu\left(\nabla\times\vec{H}\right)$$

$$=-j\omega\mu\left(j\omega\varepsilon\vec{E}\right)=\omega^2\mu\varepsilon\vec{E}$$

$$k\equiv\omega\sqrt{\mu\varepsilon}=\frac{\omega}{u_p}=\frac{2\pi}{\lambda}$$

$$\Rightarrow\nabla^2\vec{E}+k^2\vec{E}=0$$

Wave vector, propagation constant

Frequency-domain wave equation-(2)

- Take curl of Eq. (3) $\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$

$$\nabla \times \nabla \times \vec{H} = -\nabla^2 \vec{H}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

$$= \nabla \times \left(j\omega\varepsilon\vec{E} \right) = j\omega\varepsilon\left(\nabla \times \vec{E} \right)$$

$$= j\omega\varepsilon\left(-j\omega\mu\vec{H} \right) = \omega^2 \mu\varepsilon\vec{H}$$

$$\Rightarrow \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

Wave vector, wavelength, frequency

- Wave vector, propagation constant $\nabla^2 \vec{E} + k^2 \vec{E} = 0$
  - In a medium, **wavelength changes**, but not frequency

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu\varepsilon_0\varepsilon_r}, \qquad n \equiv \sqrt{\varepsilon_r}$$

$$= \frac{\omega}{u_p} = \frac{\omega}{c/n} = \frac{2\pi}{\lambda} = \frac{2\pi}{\lambda_0/n} = nk_0$$

But usually it's **n(ω)** ⇔ dispersion

Phasor-domain: compared to T-Lines

- High similarities

Homogeneous
Helmholtz's equations

$$\frac{d^2}{dz^2}V(z) + \beta^2 V(z) = 0$$

$$\frac{d^2}{dz^2}I(z) + \beta^2 I(z) = 0$$

$$\beta = \omega\sqrt{LC}$$

$$\nabla^2 \vec{E}(\vec{r}) + k^2 \vec{E}(\vec{r}) = 0$$

$$\nabla^2 \vec{H}(\vec{r}) + k^2 \vec{H}(\vec{r}) = 0$$

$$k = \omega\sqrt{\mu\varepsilon}$$

Waves in lossy medium: complex permittivity

- If the medium is conducting ($\sigma \neq 0$), the presence of $\vec{E}$ results in conduction currents $\vec{J} = \sigma\vec{E}$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \times \vec{H} = \sigma\vec{E} + j\omega\varepsilon\vec{E}$$

$$= j\omega\left(\frac{\sigma}{j\omega} + \varepsilon\right)\vec{E} = j\omega\varepsilon_c\vec{E}$$

$$\boxed{\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega}}$$ …complex permittivity

Complex permittivity → complex wave number

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

Loss tangent

$$\tan\delta_c = \frac{\sigma}{\omega\varepsilon}$$

$$k_c = \omega\sqrt{\mu\varepsilon_c} = \omega\sqrt{\mu\varepsilon\left(1 - j\tan\delta_c\right)}$$

$$k_c = k' - jk''$$

→ Loss!!!

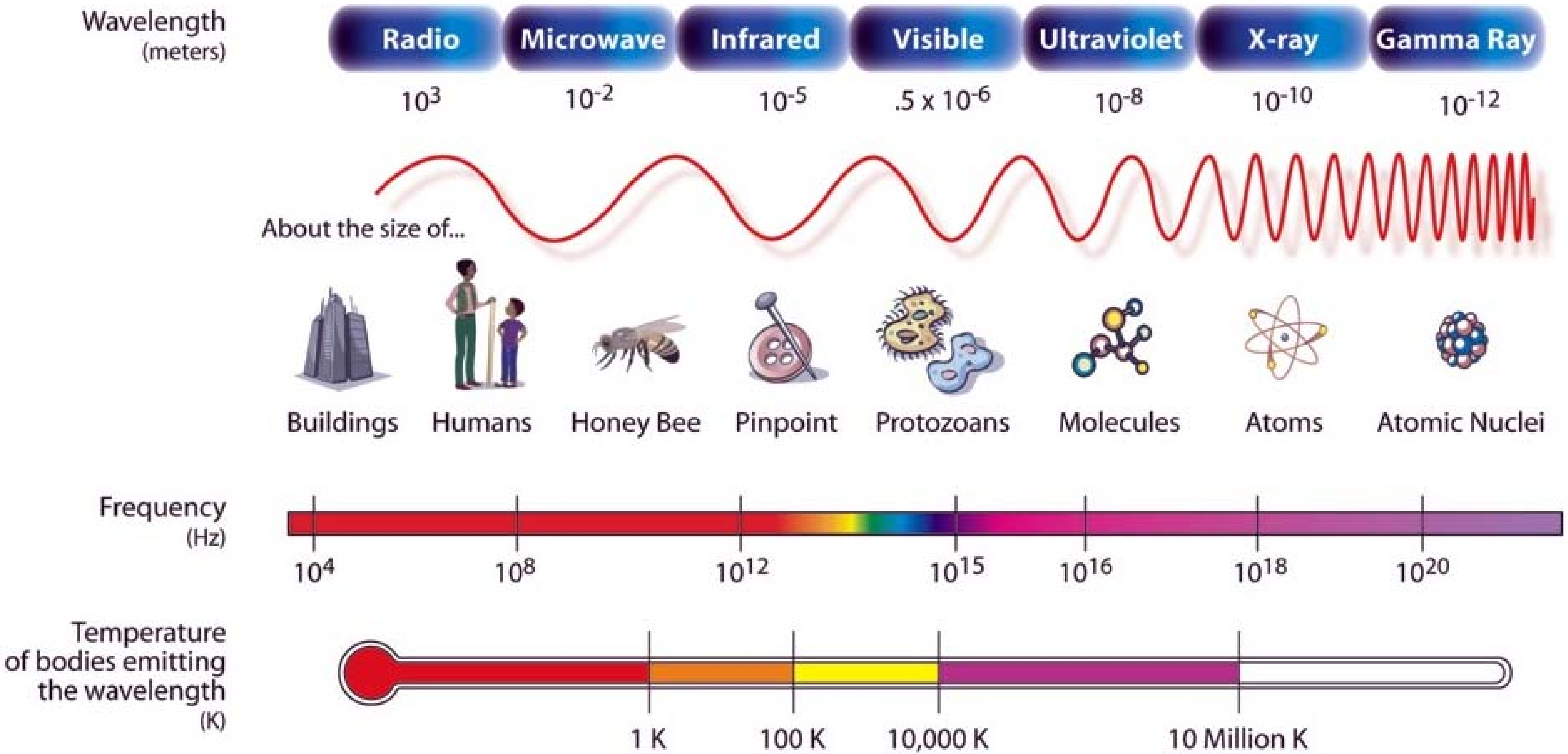
Frequency dependent!

If $\tan\delta_c \ll 1, \Rightarrow$ a dielectric

If $\tan\delta_c \gg 1, \Rightarrow$ a conductor

The EM spectrum

- Can all be calculated by Maxwell's equations



- Inhomogeneous wave equation
  - Time-varying e-dipole ➔ accelerating charge
  - Wave/light generation

Induced polarization in dielectric media

- Induced **time-varying** electric dipoles

$$\vec{D}(\vec{r},t) = \varepsilon_0 \vec{E}(\vec{r},t) + \vec{P}(\vec{r},t)$$

$$\nabla \times \nabla \times \vec{E} = -\nabla^2 \vec{E} = -\frac{\partial}{\partial t}(\nabla \times \vec{B})$$

$$\nabla \times \vec{B} = \mu_0 \frac{\partial \vec{D}}{\partial t} \Rightarrow \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{\partial \vec{P}}{\partial t}$$

This extra term gives the **Inhomogeneous Wave Equation**:

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

Inhomogeneous wave equation

- Time-varying e-dipole ➔ wave source term
- Accelerating charges

$$\vec{P}(\vec{r},t) = Nq\vec{x}_q(\vec{r},t)$$

$\vec{x}_q(t)$: separation of the charges

$$\nabla^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial^2\vec{P}}{\partial t^2} = \mu_0 Nq\frac{\partial^2\vec{x}_q}{\partial t^2}$$

But $\partial^2\vec{x}_q/\partial t^2$ is just the charge acceleration!

So it's **accelerating** charges that emit light!

Time-varying e-dipole

- Time instants
- Animation